

Interacting Gauge-Fluid system

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Abstract

A gauge-fluid relativistic model where a non-isentropic fluid is coupled to a dynamical Maxwell ($U(1)$) gauge field, has been studied. We have examined in detail the structures of energy momentum tensor, derived from two definitions, *ie.* the canonical (Noether) one and the symmetric one. In the conventional equal-time formalism, we have shown that the generators of the spacetime transformations obtained from these two definitions agree, modulo the Gauss constraint. This equivalence in the physical sector has been achieved only because of the dynamical nature of the gauge fields. Subsequently we have explicitly demonstrated the validity of the Schwinger condition. A detailed analysis of the model in lightcone formalism has also been done where several interesting features are revealed.

1 Introduction

Hydrodynamics is one of the earliest developed applied sciences [1] but in recent years, especially after the advent of AdS/CFT and subsequent fluid/gravity correspondence[2], its relevance is being appreciated in theoretical physics. From a modern high energy physics perspective, the canonical theory for relativistic perfect isentropic fluid was developed in [3], with special emphasis on symmetry aspects of the theory. On the one hand, hydrodynamics can be viewed as a universal description of long wavelength physics that deals with low energy effective degrees of freedom of a field theory, classical or quantum. Intuitively the success of fluid models rests on the expectation that at sufficiently high energy densities local equilibrium is attained in an

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interacting field theory and local inhomogeneities are smoothed out so that one is allowed to work with continuous fluid degrees of freedom. Indeed, the classical version of ideal fluid theory is a conformal field theory and this property can be exploited in AdS/CFT correspondence. On the other hand, and more interestingly, exploiting the fluid/gravity correspondence there is hope of deriving a theory of non-ideal fluid and even fluid in the presence of turbulence, based on first principles. This is because, the non-ideal fluid, being a strongly coupled one, can be dual to a weakly coupled gravity theory, again thanks to AdS/CFT, (and subsequently fluid/gravity), correspondence. The role of symmetries and their implications in fluid systems is quite crucial in this set up.

The present work is a continuation of our earlier work [4] where we presented a systematic and detailed analysis of an ideal relativistic isentropic fluid interacting with an external gauge field in the Hamiltonian framework. There are two extensions in the present paper. First, we now consider a non-isentropic fluid. Secondly, and more importantly, the present work deals with the full interacting theory where the gauge field is also dynamical. This additional input yields new interesting results and puts the interacting fluid model in a clearer perspective. As is well-known, a Lagrangian formulation of fluid dynamics is plagued with obstructions due to the presence of a Casimir operator - the vorticity (for a modern perspective see [3]). The problem can be cured by the introduction of Clebsch variables [5] designed in such a way that the vorticity becomes a surface contribution and does not obstruct the Lagrangian formulation. Furthermore, in this scheme the continuity equation of mass conservation becomes inherent in the model [7]. Following the work of two of us [8, 9] we have introduced fluid entropy and hence study non-isentropic fluid dynamics. It should be emphasized that our formalism is different from the existing works on fluid in the presence of electromagnetic interactions [10] that are essentially Hamiltonian in nature and do not provide a Lagrangian scenario. Moreover, the presentation in light cone coordinates is new.

The lightcone analysis has been provided in detail, primarily because of its role in topical concepts of non-relativistic AdS/CFT and holography [6] and also because of the non-trivial theoretical aspects of a relativistic theory itself in lightcone framework. We have compared and contrasted the results with our previous observations in [4] that dealt with perfect fluid model (without gauge interactions) in lightcone. Lightcone or Infinite Momentum Frame was introduced long ago in the context of formulating bound states of quarks and gluons in relativistic QCD [11] (for a review see [12]). In recent years lightcone quantization has reappeared strongly in the work of Son [6] who has exploited it in non-relativistic generalization of AdS/CFT and holographic principles [13].

As we have discussed in our earlier work [4], there are two distinct forms of the stress tensor based on two conventional definitions. The canonical $T_{\mu\nu}$ is obtained via Noether prescription and the symmetric $\Theta_{\mu\nu}$ is obtained by metric variation. For the free theory both definitions agree. However, in the presence of interaction, $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ do not match. Here we have been able to demonstrate that there is no inconsistency regarding this mismatch. The point is

that the physically relevant quantities are the integrated versions of $T_{\mu\nu}$ (or $\Theta_{\mu\nu}$) which define the various space time generators. Interestingly, the integrated versions of $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ agree, modulo terms proportional to the Gauss constraint. Thus these are gauge equivalent. Hence, in the physical subspace, the two definitions of the generators agree. Indeed this has been possible only because of the dynamical nature of the gauge field which brings about new constraints in the theory, in particular the Gauss law. This constraint, incidentally, did not appear for non-dynamical gauge fields. We have also discussed in detail the question of validity of the Schwinger condition [14], a hallmark of a consistent relativistic theory, in the present fluid-gauge model.

The paper is organized as follows: In section 2 the relativistic fluid model interacting with a dynamical $U(1)$ gauge field is formulated in terms of Clebsch variables in equal-time coordinate. The basic algebra between the degrees of freedom can be directly read off since the symplectic structure in the fluid sector is first order in nature. The stress tensors obtained by Noether's prescription and metric variation are shown to be conserved. Their integrated versions, defining the space-time transformation generators, are shown to be gauge equivalent. The Schwinger condition is also verified in equal time coordinate. In section 3 the lightcone analysis is performed. The first interesting observation is that even though the symplectic structure in the gauge sector is first order in nature, the algebra between the gauge field variables cannot be simply read off. We have employed Dirac's [17] scheme of Hamiltonian constraint analysis to derive the algebra. The transformation laws and conservation principles involving the stress tensor in light-cone formalism are studied. The paper ends with our conclusions and future prospects in section 4.

2 Relativistic, nonisentropic fluid mechanics in equal-time coordinates

Let us quickly recapitulate the non-isentropic fluid theory in Eulerian approach. Construction of the fluid Lagrangian requires the introduction of Clebsch variables [5, 7] $\theta, \alpha, \beta, \gamma, S$. The following combination [8, 9, 18, 19],

$$a_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta + \gamma \partial_\mu S \quad (1)$$

appears in the fluid Lagrangian as given below,

$$\mathcal{L} = -\eta^{\mu\nu} j_\mu a_\nu - f; \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (2)$$

We identify S as the entropy. The generalized scalar potential function $f(\sqrt{j^\mu j_\mu})$ dictates the dynamics. In the Eulerian description of relativistic fluid the dynamical variables are the matter density j^0 and the currents $j^i, i = 1, 2, 3$ that satisfy the conservation law,

$$\partial_\mu j^\mu = 0. \quad (3)$$

In fact the Clebsch variables are auxiliary variables that enforce the correct dynamics of j_μ . From the expanded form of the Lagrangian (2), (with $j^\mu j_\mu = n^2$, a relativistic scalar, and identifying $\rho = j^0$ as the density),

$$\mathcal{L} = -\rho\partial_0\theta - j^i\partial_i\theta - \rho\alpha\partial_0\beta - j^i\alpha\partial_i\beta - \rho\gamma\partial_0S - j^i\gamma\partial_iS - f(n), \quad (4)$$

it is straightforward to show that the current conservation law (3) follows from the θ -equation of motion.

Let us now posit the relativistic version of a fully interacting model of a fluid and a dynamical $U(1)$ gauge field as,

$$\mathcal{L} = -\eta^{\mu\nu}j_\mu(a_\nu - A_\nu) - f - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}; \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (5)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. This is a natural extension of our previous work [4] where we considered an isentropic fluid (without the entropy term) and treated the gauge field as external.

Variations of $\alpha, \beta, \gamma, S, \rho(=j^0), j^\mu, A_\mu$ yield the equations of motion,

$$j^\mu\partial_\mu\alpha = 0, \quad (6)$$

$$j^\mu\partial_\mu\beta = 0, \quad (7)$$

$$j^\mu\partial_\mu S = 0 \quad (8)$$

$$j^\mu\partial_\mu\gamma = 0 \quad (9)$$

$$\dot{\theta} + \alpha\dot{\beta} + \gamma\dot{S} + \frac{\rho}{n}f'(n) = 0. \quad (10)$$

$$j_\mu = -\frac{n}{f'(n)}(a_\mu - A_\mu) = -\frac{n}{f'(n)}(\partial_\mu\theta + \alpha\partial_\mu\beta + \gamma\partial_\mu S - A_\mu). \quad (11)$$

$$j_\beta = -\partial^\alpha F_{\alpha\beta} \quad (12)$$

It is easy to see that current conservation (3) also follows from (12). Due to the presence of this conservation, the action corresponding to (5) is invariant under the gauge transformation,

$$A_\mu \rightarrow A_\mu + \partial_\mu\Lambda, \quad (13)$$

exactly as happens in electrodynamics. This similarity persists further by noting that, as in electrodynamics, there occurs a Gauss constraint which is given by the time component of (12)

,

$$\partial_i\pi_i - j_0 = \partial_i\pi_i - \rho = 0, \quad (14)$$

where $\pi_i = \frac{\partial\mathcal{L}}{\partial A^i} = F_{i0}$ is the momentum conjugate to A^i . The Gauss constraint is the generator of the gauge transformation (13) and defines the physical subspace as

$$(\partial_i\pi_i - \rho) | \Psi \rangle_{Physical} = 0. \quad (15)$$

From (4) we can identify three independent canonical pairs (ρ, θ) , $(\alpha\rho, \beta)$ and $(\rho\gamma, S)$. The fundamental brackets, compatible with the above canonical pairs, follow from the symplectic structure,

$$\begin{aligned}\{\rho(x), \theta(y)\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{\alpha(x), \theta(y)\} = -\frac{\alpha}{\rho}\delta(\mathbf{x} - \mathbf{y}), \quad \{\alpha(x), \beta(y)\} = \frac{\delta(\mathbf{x} - \mathbf{y})}{\rho}; \\ \{\gamma(x), S(y)\} &= \frac{\delta(\mathbf{x} - \mathbf{y})}{\rho}, \quad \{\gamma(x), \theta(y)\} = -\frac{\gamma}{\rho}\delta(\mathbf{x} - \mathbf{y}),\end{aligned}\tag{16}$$

with all remaining brackets vanishing.

We now concentrate on the structure of the energy-momentum tensor. Conventionally there are two parallel definitions. One of these is the symmetric energy-momentum tensor,

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S}{\partial g^{\mu\nu}},\tag{17}$$

that is obtained by generalizing (5) to a curved spacetime which amounts to replacing $\eta^{\mu\nu}$ by $g^{\mu\nu}$, varying $g^{\mu\nu}$ and finally reverting back to flat spacetime with the replacement of $g^{\mu\nu}$ by $\eta^{\mu\nu}$ in (17).

On the other hand, the canonical energy-momentum tensor is obtained via Noether prescription,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \theta)}\partial_\nu \theta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \beta)}\partial_\nu \beta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \alpha)}\partial_\nu \alpha + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \rho)}\partial_\nu \rho + \frac{\partial \mathcal{L}}{\partial(\partial^\mu S)}\partial_\nu S + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\lambda)}\partial_\nu A^\lambda - \eta_{\mu\nu}\mathcal{L}\tag{18}$$

Both the definitions have their utilities. Both are conserved. $T_{\mu\nu}$ is designed to manifestly generate correct space-time transformations of the field variables but it is not symmetric (and can be improved by Belinfante prescription) whereas $\Theta_{\mu\nu}$ is manifestly symmetric but its ability to generate appropriate space time transformation is not transparent. In simple cases these expressions agree as is natural but there are subtleties involved in the fluid system under consideration. We emphasize that these issues have not been studied so far but become crucial for the consistency of the fluid model.

In our interacting fluid model, the canonical energy-momentum tensor is given by (2),

$$T_{\mu\nu} = -j_\mu(\partial_\nu \theta + \alpha\partial_\nu \beta + \gamma\partial_\nu S) - F_{\mu\sigma}\partial_\nu A^\sigma - \eta_{\mu\nu}\mathcal{L}.\tag{19}$$

This tensor is conserved. To show this explicitly, we exploit current conservation and other equations of motion to find,

$$\begin{aligned}\partial^\mu T_{\mu\nu} &= \partial^\mu \{-j_\mu(\partial_\nu \theta + \alpha\partial_\nu \beta + \gamma\partial_\nu S) - F_{\mu\sigma}\partial_\nu A^\sigma - \eta_{\mu\nu}\mathcal{L}\} \\ &= (\partial_\nu j^\mu)(a_\mu - A_\mu) + \partial_\nu f(n) = 0\end{aligned}\tag{20}$$

where the final step is obtained on using (11).

On the other hand the symmetric energy momentum tensor is derived from (17) as,

$$\Theta_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{j_\mu j_\nu}{n}f' - F^\beta{}_\nu F_{\beta\mu}.\tag{21}$$

This is also conserved by applying the various equations of motion,

$$\partial^\mu \Theta_{\mu\nu} = 0.$$

Now $\Theta_{\mu\nu}$ produces the Hamiltonian density

$$\begin{aligned} \Theta_{00} &= -\mathcal{L} + \frac{j_0 j_0}{n} f' - F^j{}_{00} F_{j0} \\ &= -\mathcal{L} - \rho(\partial_0 \theta + \alpha \partial_0 \beta + \gamma \partial_0 S) + \rho A_0 - F^j{}_{00} F_{j0}. \end{aligned} \quad (22)$$

Also $T_{\mu\nu}$ in (2) gives rise to the canonical Hamiltonian density,

$$T_{00} = -\rho(\partial_0 \theta + \alpha \partial_0 \beta + \gamma \partial_0 S) - F_{0\sigma} \partial_0 A^\sigma - \mathcal{L}. \quad (23)$$

Let us compute the difference between these two Hamiltonian densities,

$$\begin{aligned} T_{00} - \Theta_{00} &= -F_{0i} \partial_0 A^i + F^j{}_{00} F_{j0} - \rho A_0 \\ &= -\pi_i \partial_0 A_i - \pi_i^2 - \rho A_0 = -\pi_i (\partial_i A_0 - \pi_i) - \pi_i^2 - \rho A_0 \\ &= -\pi_i \partial_i A_0 - \rho A_0. \end{aligned} \quad (24)$$

which is obviously nonvanishing. However the physically relevant object is the integrated version corresponding to the Hamiltonian, that acts as the generator of time translation. This difference in the two expressions of Hamiltonians is found to be,

$$\begin{aligned} \int d^3x (T_{00} - \Theta_{00}) &= - \int d^3x (\pi_i \partial_i A_0 + \rho A_0) \\ &= \int d^3x A_0 (\partial_i \pi_i - \rho). \end{aligned} \quad (25)$$

which is proportional to the Gauss law. Hence, on the constraint surface (or equivalently the physical manifold) these two expressions are identical. This result should be contrasted with our previous observation in [4] where also this mismatch was noted but since the gauge field was not dynamical there was no Gauss law constraint and this mismatch persisted. Clearly the kinetic part of the gauge field, which is the Maxwell term, rounds off the theory nicely. But it is still necessary to check if the same property holds for the other important components of the stress tensor.

Let us consider the momentum density. The relevant expressions are,

$$T_{0i} = -\rho(\partial_i \theta + \alpha \partial_i \beta + \gamma \partial_i S) - F_{0j} \partial_i A^j = -\rho(\partial_i \theta + \alpha \partial_i \beta + \gamma \partial_i S) - \pi_j \partial_i A^j, \quad (26)$$

$$\Theta_{0i} = \frac{j_0 j_i}{n} f' - F^\beta{}_{i0} F_{\beta 0} = -\rho(\partial_i \theta + \alpha \partial_i \beta + \gamma \partial_i S) + \rho A_i - F^k{}_{i0} F_{k0}, \quad (27)$$

with the difference

$$T_{0i} - \Theta_{0i} = -\rho A_i - \pi_j \partial_i A^j + \pi_j (\partial_i A^j - \partial_j A^i) = -(\rho A_i + \pi_j \partial_j A_i). \quad (28)$$

Once again integration of the above result yields

$$\int d^3x (T_{0i} - \Theta_{0i}) = \int d^3x A_i (\partial_i \pi_i - \rho), \quad (29)$$

indicating that the total momenta in the two definitions are equal modulo the first class (Gauss) constraint. Incidentally, these integrated expressions act as spatial translation generator. Exploiting the covariant notation, the combination of (25) and (29) is written in a compact form,

$$\int d^3x (T_{0\mu} - \Theta_{0\mu}) = \int d^3x A_\mu (\partial_i \pi_i - \rho), \quad (30)$$

which vanishes on the physical subspace.

It is possible to continue this analysis for the angular momentum operator. From Noether's definition, this is given by,

$$M_{ij}^N = \int (x_i T_{0j} - x_j T_{0i} - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \Sigma_{ij}^{\lambda\sigma} A_\sigma) d^3x \quad (31)$$

where the spin tensor is defined as,

$$\Sigma_{\alpha\beta}^{\lambda\sigma} = g_\alpha^\lambda g_\beta^\sigma - g_\beta^\lambda g_\alpha^\sigma. \quad (32)$$

We therefore obtain,

$$M_{ij}^N = \int (x_i T_{0j} - x_j T_{0i} - \pi_i A_j + \pi_j A_i) d^3x$$

The angular momentum, following from the symmetric tensor (17), is given by

$$M_{ij}^S = \int (x_i \Theta_{0j} - x_j \Theta_{0i}) d^3x$$

Using (27) and (28) it is seen that the difference between these expressions vanishes, modulo terms proportional to the Gauss constraint,

$$M_{ij}^N - M_{ij}^S = \int d^3x (x_i A_j - x_j A_i) (\partial_k \pi_k - \rho). \quad (33)$$

Thus on the physical subspace, the expressions for angular momenta are identical, as happened for the space-time translation generators discussed earlier.

Similarly, the difference in the structures of the boost generators can also be discussed. From Noether's definition, the boost is given by,

$$M_{0i}^N = \int (x_0 T_{0i} - x_i T_{00} - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \Sigma_{0i}^{\lambda\sigma} A_\sigma) d^3x \quad (34)$$

From (32) it follows,

$$M_{0i}^N = \int (x_0 T_{0i} - x_i T_{00} - \pi_i A_0) d^3x. \quad (35)$$

On the other hand, the definition of boost following from the symmetric tensor (17) is,

$$M_{0i}^S = \int (x_0 \Theta_{0i} - x_i \Theta_{00}) d^3x. \quad (36)$$

Once again the difference is just proportional to the Gauss constraint,

$$M_{0i}^N - M_{0i}^S = \int d^3x (x_0 A_i - x_i A_0) (\partial_k \pi_k - \rho) \quad (37)$$

In fact (33) and (37) maybe combined to yield a covariant structure,

$$M_{\mu i}^N - M_{\mu i}^S = \int d^3x (x_\mu A_i - x_i A_\mu) (\partial_k \pi_k - \rho). \quad (38)$$

Indeed, the above exercise is non-trivial since it underlines the importance of introducing the Maxwell gauge field kinetic term and also establishes the spacetime symmetries of the fully interacting relativistic fluid model in a robust way. This explicit demonstration was absent in previous literature.

Schwinger condition:

In its simplest form, the Schwinger covariance condition relates the equal-time energy density commutator to the momentum density,

$$[\Theta_{00}(x), \Theta_{00}(x')] = (\Theta_{0i}(x) + \Theta_{0i}(x')) \partial_i \delta(x - x'). \quad (39)$$

For some quantum field theoretical applications see [15], this is referred to as Dirac-Schwinger condition [16]. Validity of this condition in a quantum field theory ensures that the theory is relativistically covariant. However, it can play an important role in field theories even in non-relativistic scenario [4, 8].

Let us now concentrate on the Schwinger condition for the present model. In our previous paper [4] we have demonstrated the validity of the Schwinger condition for the non-interacting fluid model. The situation is more complicated here because the gauge fields being dynamical satisfy a canonical Poisson algebra $\{A_i(\mathbf{x}), \pi^j(\mathbf{y})\} = \delta_i^j \delta(\mathbf{x} - \mathbf{y})$. We need to compute the following bracket,

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = \{j^i(\partial_i \theta + \alpha \partial_i \beta - A_i) + f + \frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi_i^2|_x, j^k(\partial_k \theta + \alpha \partial_k \beta - A_k) + f + \frac{1}{4} F^{lm} F_{lm} + \frac{1}{2} \pi_k^2|_y\}. \quad (40)$$

After a long but straightforward calculation, we arrive at the result,

$$\begin{aligned} \{\Theta_{00}(x), \Theta_{00}(y)\} &= [(-\rho(\partial_i \theta + \alpha \partial_i \beta - A_i) + F_{ik} \pi^k|_x) + (-\rho(\partial_i \theta + \alpha \partial_i \beta - A_i) + F_{ik} \pi^k|_y)] \partial_i^x \delta(x - y) \\ &= (\Theta_{0i}(x) + \Theta_{0i}(y)) \partial_i^x \delta(x - y). \end{aligned} \quad (41)$$

This ensures the validity of the Schwinger condition in the fully interacting fluid-Maxwell theory.

3 Light cone analysis

Light-cone (or light front form of) quantization (LCQ) was introduced very early with two principal motivations: as a computational tool for bound state solutions in QCD to represent

hadrons as bound states of quarks and gluons in a relativistic framework and also to utilize computers in quantum field theory calculations. (See [12] for an early review.) In fact the convenience of LCQ was pointed out by Dirac [20] as an alternative to equal time quantization where the lightcone coordinates are defined as as [21] $\{x^+, x^-, \bar{x}\}$, where

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3); \quad \bar{x} \equiv x^a = x^1, x^2. \quad (42)$$

Here x^+ plays the role of time and \bar{x} are referred as transverse coordinates. The non-vanishing metric components are,

$$g^{+-} = g^{-+} = 1; \quad g^{ab} = \delta^{ab}, \quad a, b = 1, 2. \quad (43)$$

In the context of QCD a related framework, known as Infinite Momentum Frame, was initiated [11] to explain Bjorken scaling in scattering phenomena. The physical meaning of this correspondence is that measurements made by an observer moving at infinite momentum is equivalent to making observations with speed being close to the speed of light and this corresponds to the front form where measurements are made along the front of a light wave.

Coming back to recent times LCQ has generated tremendous amount of interest after the work of Son [6] who formulated a model that represented the experimentally demonstrated trapping of cold atoms at Feshbach resonance, thereby introducing the concept of non-relativistic holographic principle in AdS/CFT correspondence. It is important to note that in light cone variables a second order system, (in terms of time derivative), such as Klein Gordon, is changed to a first order system such as Schrodinger. But precisely this algebraic manipulation drastically alters the Hamiltonian structure of the system because the converted first order system turns out to be a constraint system with a non-canonical symplectic structure and reduced number of degrees of freedom. We will explicitly demonstrate that there are subtleties involved in the Hamiltonian analysis since the lightcone coordinate system is qualitatively distinct from the conventional equal time coordinate framework. At this point it is worthwhile to recall our earlier work [4] where, for the first time, a detailed lightcone analysis of the free fluid system was performed. There [4] it was observed that the symplectic structure in lightcone coordinate did not differ from the one in equal time coordinate, the reason being that the free fluid model was a first order system even in equal time coordinate. However, the difference between the two frameworks was manifest in *eg.* Schwinger condition where the spatial coordinates, x_- and transverse ones \bar{x} , were clearly separated into different sectors. In the present work, where we consider the fully interacting fluid-Maxwell theory, the situation becomes much more serious since the Maxwell gauge sector is quadratic in nature and upon LCQ leads to complications that puts a question mark on the validity of the Schwinger condition. This is not surprising since, in the hamiltonian framework, LCQ even for a simple massless scalar theory involves subtleties and complications. However, we emphasize, that the total energy of the system remains conserved in LCQ.

Massless scalar:

The intricacies of LCQ can be seen in the simplest of models, that of a massless scalar field. It also helps in setting up the notation and introduce some basic formulae. The Lagrangian

$$L = \int d^4x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (44)$$

generates the equation of motion,

$$\partial_\mu \partial^\mu \phi = 0. \quad (45)$$

The same equation is recovered from the Hamiltonian

$$H = \int d^3x \mathcal{H}(x) = \int d^3x \frac{1}{2} (\pi^2 + \partial_i \phi \partial_i \phi), \quad (46)$$

as Hamilton's equation of motion where the equal-time canonical algebra $\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y})$, $\{\phi(\mathbf{x}), \phi(\mathbf{y})\} = \{\pi(\mathbf{x}), \pi(\mathbf{y})\} = 0$ is used.

It is now straightforward to compute the bracket between the energy densities $\{\mathcal{H}(x), \mathcal{H}(y)\}$ to yield,

$$\left\{ \frac{1}{2} (\pi^2 + \partial_i \phi \partial_i \phi)(x), \frac{1}{2} (\pi^2 + \partial_i \phi \partial_i \phi)(y) \right\} = ((\pi \partial_i \phi)(x) + \pi \partial_i \phi)(y)) \partial_i \delta(\mathbf{x} - \mathbf{y}). \quad (47)$$

The above equation amounts to,

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = (\mathcal{P}_i(x) + \mathcal{P}_i(y)) \partial_i \delta(\mathbf{x} - \mathbf{y}), \quad (48)$$

thereby yielding the Schwinger condition where $\mathcal{P}_i = \pi \partial_i \phi$ is defined as the momentum density.

The same Lagrangian, now expressed in lightcone coordinates,

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{1}{2} \partial_i \phi \partial_i \phi \quad (49)$$

generates the equation of motion,

$$2\partial_+ \partial_- \phi = \partial_i \partial_i \phi, \quad (50)$$

which is identical to (45). However, recovering (50) in Hamiltonian formalism is more complicated since the lightcone Lagrangian (49) possesses a constraint

$$\Omega(x) \approx \pi(x) - \partial_- \phi(x), \quad (51)$$

where the momentum is defined as $\pi = (\partial \mathcal{L})/(\partial(\partial_+ \phi)) = \partial_- \phi$. This feature is typical of second order systems when expressed in lightcone variables. The constraint $\Omega(x)$ turns out to be second class in the Dirac sense [17] since the constraint bracket is non-vanishing,

$$\{\Omega(x), \Omega(y)\} = 2\partial_- \delta(x_- - y_-) \delta(\bar{x} - \bar{y}). \quad (52)$$

Canonical symplectic structure is used to derive (52). Defining $\partial_- \epsilon(x_- - y_-) = \delta(x_- - y_-)$ one obtains the inverse of the RHS of (52) as

$$[\{\Omega(x), \Omega(y)\}]^{-1} = \frac{1}{2} \epsilon(x_- - y_-) \delta(\bar{x} - \bar{y}). \quad (53)$$

Following the well-known prescription one computes the non-canonical symplectic structure (or Dirac bracket [17])

$$\begin{aligned}\{\phi(x_+, x_-, \bar{x}), \phi(y_+, y_-, \bar{y})\} &= - \int dz_1 dz_2 \{\phi(x), \Omega(z_1)\} [\{\Omega(z_1), \Omega(z_2)\}]^{-1} \{\Omega(z_2), \phi(y)\} \\ &= \frac{1}{2} \epsilon(x^- - y^-) \bar{\delta}(\bar{x} - \bar{y}).\end{aligned}\quad (54)$$

Note the non-local nature of the lightcone symplectic structure. Together with the Hamiltonian

$$H = \int dy^- d\bar{y} \partial_i \phi \partial_i \phi \quad (55)$$

and the algebra (3), we compute $\partial_+ \phi$,

$$\partial_+ \phi = \partial_{\bar{x}}^i \int dy^- \partial_i \phi(x_+, \bar{x}, y^-) \frac{1}{2} \epsilon(x^- - y^-). \quad (56)$$

Furthermore, on differentiating both sides by ∂_- the non-locality is removed and one recovers the correct equation of motion (50). This clearly underlines the fact that lightcone framework, while reproducing the equal time equation of motion (45), is qualitatively distinct from the equal time framework with a reduced number of degrees of freedom due to the constraint. The original second order system is converted to first order.

To derive the energy conservation principle from the covariant form of the symmetric energy momentum tensor for the massless scalar,

$$\Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \mathcal{L} \eta^{\mu\nu} \quad (57)$$

we first write down the different lightcone components as,

$$\Theta^{--} = (\partial_+ \phi)^2, \quad \Theta^{i-} = -\partial_i \phi \partial_+ \phi, \quad \mathcal{H} \equiv \Theta^{+-} = \partial_i \phi \partial_i \phi \quad (58)$$

where Θ^{+-} is identified with the Hamiltonian density \mathcal{H} .

Let us now calculate the time derivative of Θ^{+-} ,

$$\partial_+ \Theta^{+-} = \{\mathcal{H}(x), H\} = \left\{ \frac{1}{2} \partial_i \phi(x) \partial_i \phi(x), \int dy^- d\bar{y} \frac{1}{2} \partial_i \phi \partial_i \phi \right\}. \quad (59)$$

Using the algebra (3) we find

$$\begin{aligned}& \left\{ \frac{1}{2} \partial_i \phi(x) \partial_i \phi(x), \int dy^- d\bar{y} \frac{1}{2} \partial_j \phi(y) \partial_j \phi(y) \right\} \\ &= \frac{1}{2} (\partial_i \phi)(x) \partial_i^x \int dy^- d\bar{y} (\partial_j \partial_j \phi)(y) \epsilon(x^- - y^-) \delta^2(\bar{x} - \bar{y}).\end{aligned}$$

Exploiting the lightcone equation of motion (50) we obtain,

$$\partial_+ \Theta^{+-} = \partial_i \phi \partial_i \partial_+ \phi = -\partial_- [\partial_+ \phi(x)^2] + \partial_i [\partial_+ \phi(x) (\partial_i \phi(x))], \quad (60)$$

where the final step is obtained by again using (50). Comparing with (58) we rewrite (60) as,

$$\partial_\mu \Theta^{\mu-} = \partial_+ \Theta^{+-} + \partial_- \Theta^{--} + \partial_i \Theta^{i-} = 0. \quad (61)$$

This validates the conservation of energy. Likewise the other components of $\partial_\mu \Theta^{\mu\nu} = 0$ can be shown to hold.

Interacting fluid:

Returning to the interacting fluid model, let us rewrite the Lagrangian (5) in lightcone variables,

$$\mathcal{L} = -j^+(\partial_+\theta + \alpha\partial_+\beta - A_+) - j^-(\partial_-\theta + \alpha\partial_-\beta - A_-) - j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) - f \quad (62)$$

$$-\frac{1}{4}(2F^{+-}F_{+-} + 2F^{+i}F_{+i} + 2F^{-i}F_{-i} + f^{ij}F_{ij}). \quad (63)$$

Interestingly the symplectic structure in the fluid sector remains essentially unaffected since it was already in a first order form (in equal time framework in (5)). Hence the previous fluid algebra (16) suffices. But the constraint structure in the gauge sector is much more involved. From the definition of the conjugate momenta $\frac{\partial\mathcal{L}}{\partial(\partial_+A^\mu)} = \pi^\mu = F^{\mu+}$, we recover three primary constraints,

$$\Omega_1 = \pi^+ \approx 0, \quad \chi_a = \pi^a - F^{a+} \approx 0, \quad a = 1, 2. \quad (64)$$

The canonical Hamiltonian density is obtained following the conventional definition

$$\mathcal{H} = \frac{1}{2}(\pi^-)^2 + \frac{1}{2}F^{12}F_{12} + (\pi^-\partial_- + \pi^i\partial_i)A^-. \quad (65)$$

The Gauss constraint Ω_2 in lightcone is obtained by calculating the Poisson bracket of π^+ with the Hamiltonian,

$$\Omega_2 = \{\pi^+(x), \int dy^- d\bar{y} \mathcal{H}(y)\} = \partial_i\pi^i(x) + \partial_-\pi^-(x) + j^+(x) \approx 0. \quad (66)$$

This is just the time (+) component of the equation of motion (12). Now Ω_1, Ω_2 constitute a set of first class constraints (indicating gauge invariance of the theory) whereas χ_a turn out to be a second class set of constraints that induce the Dirac brackets,

$$\begin{aligned} \{\pi^-, A_i\} &= \frac{1}{4}\partial_i^x \epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y}), \quad \{\pi^-, \pi^-\} = -\frac{1}{4}\nabla^2(x)\epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y}), \\ \{A^i, A^j\} &= \frac{1}{4}\epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y})\delta_{ij}. \end{aligned} \quad (67)$$

For computational details the reader is encouraged to consult [21].

First of all we ensure that our earlier observation regarding the equality of the two definitions of the (integrated) energy momentum tensor modulo Gauss constraint remains valid in lightcone. For this we we explicitly write down the expressions for $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ respectively,

$$\begin{aligned} T^{+-} &= \frac{1}{2}(\pi^-)^2 + \frac{1}{2}F_{12}F^{12} + (\pi^-\partial_- + \pi^i\partial_i)A^- + f \\ &\quad + j^-(\partial_-\theta + \alpha\partial_-\beta) + j^i(\partial_i\theta + \alpha\partial_i\beta) - j^\mu A_\mu, \\ T^{+i} &= -j^+(\partial^i\theta + \alpha\partial^i\beta) + F^{-+}\partial^i A_- - F^{+j}\partial^i A_j, \end{aligned} \quad (68)$$

$$\begin{aligned} \Theta^{+-} &= \frac{1}{2}(\pi^-)^2 + \frac{1}{2}F_{12}F^{12} + j^-(\partial_-\theta + \alpha\partial_-\beta - A_-) + j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) + f, \\ \Theta^{+i} &= -j^+(\partial^i\theta + \alpha\partial^i\beta - A^i) + F^{-+}F_-^i + F_{ij}\pi_j. \end{aligned} \quad (69)$$

It is easy to check that the following relations hold:

$$\int dy^- d\bar{y} (T^{+-} - \Theta^{+-}) = - \int dy^- d\bar{y} (\partial_i \pi^i + \partial_- \pi^- + j^+) A^-, \quad (70)$$

$$\int dy^- d\bar{y} (T^{+i} - \Theta^{+i}) = - \int dy^- d\bar{y} (\partial_i \pi^i + \partial_- \pi^- + j^+) A^i. \quad (71)$$

Integrated forms of the canonical and symmetric structures of the energy momentum tensor differ by a term proportional to the Gauss constraint (66) and hence are equal in the physical subspace in lightcone coordinates. Equations (70, 71) are the light cone analogues of the equal time relations given in (30).

The integrated energy-momentum tensors execute the spacetime translations. This naturally leads to the question as to what happens to rest of the spacetime transformations that is rotation and boosts. Since the derivation is somewhat tricky we give the details below.

Once again the basic distinction between the spatial coordinates y_- and \bar{y} comes in to play and we need to perform the calculations individually. First of all we provide integrated expressions for the 12-component of angular momentum, M_{12} , in the two definitions, derived respectively from (31) and (17),

$$M_{12}^N = \int dx^- d\bar{x} \{ (x_1 T_{-2} - x_2 T_{-1}) - (\pi_1 A_2 - \pi_2 A_1) \}, \quad (72)$$

$$M_{12}^S = \int dx^- d\bar{x} \{ (x_1 \Theta_{-2} - x_2 \Theta_{-1}) \}. \quad (73)$$

Difference between these two expressions is computed below,

$$\begin{aligned} M_{12}^N - M_{12}^S &= \int dx^- d\bar{x} \{ x_1 (T_{-2} - \Theta_{-2}) - x_2 (T_{-1} - \Theta_{-1}) - (\pi_1 A_2 - \pi_2 A_1) \} \\ &= \int dx^- d\bar{x} \{ x_1 (-j^+ A_2 + \pi^- \partial_- A_2 + \pi^i \partial_i A_2) - x_2 (-j^+ A_1 + \pi^- \partial_- A_1 + \pi^i \partial_i A_1) - (\pi_1 A_2 - \pi_2 A_1) \} \\ &= \int dx^- d\bar{x} \{ (x_1 A_2 - x_2 A_1) (\partial_- \pi^- + \partial_i \pi^i + j^+) - (\pi_2 A_1 - \pi_1 A_2) - (\pi_1 A_2 - \pi_2 A_1) \} \\ &= \int dx^- d\bar{x} \{ (x_1 A_2 - x_2 A_1) (\partial_- \pi^- + \partial_i \pi^i + j^+) \}. \end{aligned} \quad (74)$$

Integration by parts is done to obtain the last step which shows that the expressions are equal modulo Gauss constraint (66). Rest of the components of M_{+i}^N and M_{+i}^S are given by,

$$M_{+i}^N = \int dx^- d\bar{x} \{ (x_+ T_{-i} - x_i T_{-+}) - (\pi_+ A_i - \pi_i A_+) \}, \quad (75)$$

$$M_{+i}^S = \int dx^- d\bar{x} \{ (x_+ \Theta_{-i} - x_i \Theta_{-+}) \}. \quad (76)$$

Their difference turns out to be,

$$\begin{aligned} M_{+i}^N - M_{+i}^S &= \int dx^- d\bar{x} \{ x_+ (T_{-i} - \Theta_{-i}) - x_i (T_{-+} - \Theta_{-+}) - (\pi_+ A_i - \pi_i A_+) \} \\ &= \int dx^- d\bar{x} \{ (x_+ A_i - x_i A_+) (\partial_- \pi^- + \partial_i \pi^i + j^+) - (\pi_i A_+ - \pi_+ A_i) - (\pi_+ A_i - \pi_i A_+) \} \end{aligned}$$

$$= \int dx^- d\bar{x} \{ (x_+ A_i - x_i A_+) (\partial_- \pi^- + \partial_i \pi^i + j^+) \}. \quad (77)$$

Likewise, the difference among the boosts defined as,

$$M_{-i}^N = \int dx^- d\bar{x} (x_- T_{-i} - x_i T_{--} - \pi_i A_-) \quad (78)$$

and,

$$M_{-i}^S = \int dx^- d\bar{x} (x_- \Theta_{-i} - x_i \Theta_{--}) \quad (79)$$

also turns out to be proportional to the Gauss constraint,

$$M_{-i}^N - M_{-i}^S = \int dx^- d\bar{x} (x_- A_i - x_i A_-) (\partial_- \pi^- + \partial_i \pi^i + j^+) \quad (80)$$

Hence we have explicitly demonstrated that, in lightcone coordinates as well, the spacetime symmetry generators, obtained from the Noether and symmetric prescriptions, are equal modulo Gauss constraint which means that they are identical when acting on the physical subspace. It is also straightforward to establish the energy momentum conservation for the fluid gauge model in lightcone framework where the fundamental brackets provided in (16,3) need to be used. We have not given the detailed derivation since it is not very illuminating.

4 Conclusion and future prospects

Let us summarize our work. We have extended our previous work [4] in two ways: we have added the entropy term to the fluid sector and have included the Maxwell term in the gauge sector. The latter makes the gauge fields dynamical so that a fully interacting gauge-fluid theory has been considered. We have concentrated primarily on the relativistic aspect of the theory and have studied in detail the structures of energy momentum tensor, derived from two definitions, *ie.* the canonical (Noether) one and the symmetric one. In the conventional equal-time formalism, we have shown that all the spacetime symmetry generators obtained from these two definitions agree modulo the Gauss constraint. This equivalence in the physical sector has been achieved only because of the kinetic term of the gauge fields. We consider this an important finding since, in the absence of this term, this equivalence cannot be shown. Subsequently we have explicitly demonstrated the validity of the Schwinger condition in the full theory. Apart from its intrinsic appeal it also ensures that the unconventional nature of the fluid symplectic structure (with the auxiliary fluid variables) does not spoil the relativistic covariance of the model.

Another important aspect of our work is the detailed analysis of the gauge-fluid model in the lightcone formalism. Inded this lightcone analysis has several non-trivial features but unfortunately a rigorous analysis of it, especially in the context of fluid dynamics was lacking. Because of the recent interest in lightcone framework, we have carried out a detailed study. We have shown that the conservation principles are maintained. Furthermore we have explicitly demonstrated that as in the equal time case discussed here, the space time symmetry generators differ by the lightcone form of Gauss law.

What are the possible future directions? One interesting and topical problem is in the context of anomalous fluid dynamics with triangle anomalies in the form of Adler-Bell-Jackiw anomaly [22]. Of particular interest is the recent work [23] where the authors propose a field theoretic fluid gauge model to represent the relativistic hydrodynamic formulation of Abelian gauge anomaly in [24]. However, the work in [23] deals with a purely non-dynamical gauge field and we believe that it can be improved by introducing the Maxwell term in the gauge sector thereby studying the fully interacting theory as we have done here. Also the Hamiltonian analysis for the anomalous theory will be interesting both in equaltime and lightcone frameworks due to the presence of the non-trivial algebra in the gauge sector.

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